Journal of Engineering Physics and Thermophysics, Vol. 76, No. 5, 2003

MODELING OF HYPERSONIC FLOWS: INFLUENCE OF THE STARTING CONDITIONS OF THE ALGORITHM ON THE FINAL SOLUTION IN THE VICINITY OF BIFURCATION POINTS

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UDC 519.6:531.332

We investigate the influence of the starting conditions of the computational algorithm (a discrete analog of the initial data of a continuous differential problem) on the trajectory of calculation advancement in the space of solutions and on the obtained final (stationary, quasi-stationary, or nonstationary) solutions of Navier–Stokes equations for mathematical modeling of high-speed gas flows. Particular consideration is given to the analysis of the behavior of trajectories in the vicinity of the bifurcation points of branching of solutions. Questions associated with the "carbuncle" effect — a special kind of instability of a part of the shock front at a hypersonic flow over the front part of a blunt-nosed body — are discussed.

In [1], to investigate the features of the flow of the front parts of blunt bodies by a hypersonic gas stream with the excitation of additional degrees of freedom (vibrational or electronic ones), computer simulation based on numerical integration of the nonstationary system of complete Navier–Stokes equations was used. The calculations made in a wide range of diagnostic variables pointed to the existence of regions of weak and strong instabilities in gases with small values of the adiabatic index [2, 3] for finite perturbations of parameters in the incident flow. Tarnavskii et al. [4] considered a number of problems connected with the nonuniqueness of the Navier–Stokes numerical equations and investigated the bifurcation-type instability leading to the transition of the nonstationary process from one branch of the solution to another and its going to the stationary or quasi-stationary regime.

Under conditions close to the neutral instability regime [1], in [5] the nonstationary quasi-periodic process caused by the carry-over of the pulse of the external perturbation source to the stationary pattern of the hypersonic flow past the blunt front part of the body was investigated. Its basic feature is the cyclic motion of the internal shock wave (formed at the initial instant of time) from the front shock to the surface of the body and back, which leads to the pulsing character of the time dependence of aerodynamic coefficients, including that with a considerable decrease in the drag coefficient at certain stages of this cyclic process.

In [6], on the basis of numerical experiments problems of obtaining asymmetric solutions in symmetric problems with the use of symmetric algorithms were investigated and analyzed. Here, in carrying out series of calculations of the flow of bodies, in a number of ranges of diagnostic variables hysteresis phenomena were revealed: the steady-state solutions obtained strongly depended on the choice of the initial data. In this work, the related equations of nonuniqueness of numerical solutions of Euler and Navier–Stokes systems were discussed from the point of view of the influence of the initial data of the differential problem (starting conditions of the discrete algorithm) and attraction fields and attractors of the computational algorithms were determined. The common features of the solutions of all previously considered problems: the flow separation and the pulsing quasi-stationary aperiodic character of the flow were formulated.

In the present paper, which is a sequel to the investigations of [1–6]. we investigate the important, and, in some cases, even the key role of the starting conditions determining the phase portrait of the solution in the hyper-

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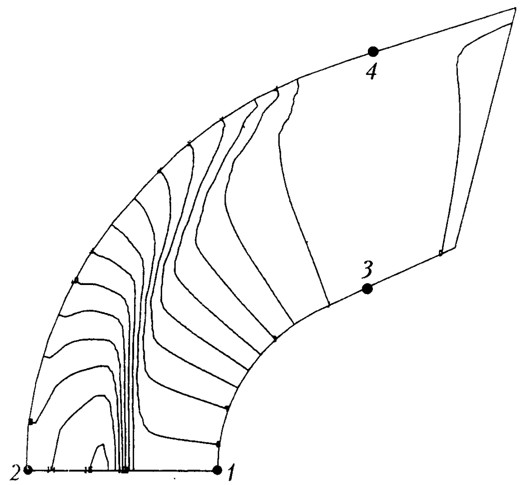

Fig. 1. Flow of a blunt body by a hypersonic gas stream. Density isolines in the disturbed flow region between the body surface and the front shock. M_∞ = 10, Re_∞ = 25,000, γ = 1.03.

space of basic parameters. The main aspect of the paper is the tracing and analysis of the trajectory of advancement of the numerical solution from the starting one to the final solution — stationary (if any), quasi-stationary (periodic or aperiodic), or nonstationary (with amplitude-restricted oscillations or a nonstationary, infinitely increasing one). Particular consideration is given to the search for a bifurcation point of the solution — the possible transition from one branch of the solution to another and to the analysis of the behavior of the solution in the vicinity of this point.

Scheme of the Computing Experiment. The front part of a blunt-nosed, heat-insulated body (sphere-cone with angle β) is flown by a hypersonic viscous heat-conducting gas stream with a set of diagnostic variables M_∞, Re_∞, Pr, ω, and γ. In the region bounded by the surface of the body, the front shock, and the outside boundary (placed in the zone where the flow is supersonic, except for the boundary layer) numerical integration of the system of Navier–Stokes equations is carried out. This system is closed by the equation of state (e.g., by the equation of a perfect gas state) and by the assignment of a particular kind of temperature dependence (e.g., a power law) of viscosity and heat-conductivity coefficients (for more details of the formulation of the problem in the algorithm used, see [2, 3, 7, 8]). The choice of the initial data of the differential problem — the starting conditions of the discrete algorithm, which concludes the mathematical formulation of the problem and actually is basic in this set of numerical experiments — was made so that answers to the following questions could be obtained:

1) If a stationary solution exists (i.e., a solution with a certain accuracy varied in the course of the experiment for checking has been found), is it unique?

2) Is it possible to obtain another stationary solution by varying the starting conditions (SCs) and, if "yes", what is the route of advancement of the solution (RAS) from the SCs to the final solution (FS)?

3) If there are no stationary regimes and only quasi- and nonstationary FSs are formed, what is the influence of the SCs on the FSs, in particular, what is the vibration amplitude and the logarithmic decrement in the case of "convergence" of FSs to some periodic or aperiodic regime?

4) What is the level of allowed perturbations introduced by the formulation of the SCs in making calculations in the vicinity of the bifurcation point — branching of the numerical solution?

5) Is the transition from one branch of the solution to another single and determinate or is at least one reverse transition of the solution possible?

6) Is, in the zone of weakly or strongly unstable FSs, the phenomenon of "carbuncle" — of "boiling" of the central part of the front shock in the vicinity of the drag line investigated in [9, 10] on close problems — observed?

Note that in the present work the possible nonuniqueness of numerical solutions is meant; investigations of bifurcations of the solutions of differential problems lie in a somewhat different plane.

The scheme of the computing experiment is shown in Fig. 1. To solve a problem with some diagnostic set of variables as SCs, the previously obtained (stationary or quasi-stationary) solution with other but close values of these variables is substituted. Because of their large number, in the present paper the principal variable quantity is the effective adiabatic index γ that permits taking into account the physicochemical processes in gas [2, 3]. With such a formulation of the problem the following process is realized: from the front shock surface to the flow region the internal shock wave formed at the initial instant of time is moving, which changes the solution. For convenience of visualization, the scale of the radial coordinate direction in Fig. 1 in the region between the body and the front shock has been magnified (20×).

In the integration domain, we chose four check points — "sensors" — at which changes in the gas dynamic functions at each instant (iteration step) of the calculated time are traced: 1 — stagnation point, 2 — point of intersection of the drag line and the front shock, 3 and 4 — points of intersection of the coordinate ray emerging from the spherical bluntness of the body perpendicular to the symmetry axis of the problem with the surface of the body and the front shock, respectively. Such an arrangement of the check points in a "square" is most convenient for analyzing the obtained information.

Results of the Computing Experiment. Because of the multiparametric character of the problem computations, their results are presented below with the following set of parameters: $\beta = 0$, $M_\infty = 10$, $Re_\infty = 25{,}000$, $Pr = 0.72$, and $\omega = 0.75$. The values of γ are varied in the domain of their definition having a physical meaning: $\gamma \in [1, 5/3]$.

All segments in Fig. 2a given below show the routes of advancement of solutions at check points 1–4 (curves 1–4 respectively) from the starting conditions with the value of $\gamma = \gamma_0$ to the final solution with $\gamma = \gamma_s$. In these figures, the evolution (the dependence on the interaction step number) of the local gas density $\rho(N)$ is analyzed. As scale values, ρ_∞, V_∞, and R are used, which makes it possible, if necessary, to convert the dimensionless parameters to dimensional ones.

Figure 2a shows the RASs from the SCs with $\gamma_0 = 1.67$ to the FS with $\gamma_s = 1.4$. For this variant, the FS is a stationary solution that had reached the steady state by $N = 600$, and at points 1 and 2 thereby the steady state had been reached much earlier. In this calculation (and in other calculations unless otherwise specified), a very small step, $\tau = 0.001$, was specially used to obtain detailed information about the RAS (generally speaking, the algorithm for obtaining stationary solutions, if any, admits much larger τ). The FS obtained is a unique solution: the use of other initial data led to one and the same FS with the only difference in the RAS. Analogous RAS patterns took place throughout the subrange of $\gamma > 1.12$: all FSs obtained were stationary and unique for any SC used in the experiment. This result was expected, since these FSs lie in the stability region (see [1]).

A different situation arises when the calculation parameters approach the boundary of neutral stability. Figure 2b shows the RAS for calculating the problem with $\gamma_s = 1.11$ and $\gamma_0 = 1.2$. The FS is already quasi-stationary with a small oscillation amplitude of the order of 0.5 and 1% at points 3 and 4, respectively, and at points 1 and 2 the FS practically had reached the steady state by the instant $N = 1000$. The attraction basin of this type of FSs is a large region of SCs: in the range of $\gamma_0 \in (1.12, 1.2)$ all FSs were quasi-stationary, with an undamped low level of oscillations about one and the same value.

Because of the small amplitude, we failed to localize the source of oscillations. There is a proposal that this source is the intersection point of the sonic line and the front shock. More definite conclusions in favor of or against this proposal are rather difficult to draw despite the fact that there is complete information on the numerical solution. This is due to the fact that along the front shock surface from point 2 to point 4 the change in the parameters and the amplification of the oscillation background are so smooth that it is difficult to single out any special gradient zone and determine it definitively as a "disturbance source." One can also make an alternative proposal that the disturbance source as such does not exist at all but the whole front shock wave oscillates weakly, and the oscillation amplitude thereby, naturally, increases downstream.

Figure 2c shows the RASs for the problem with $\gamma_s = 1.10$ and $\gamma_0 = 1.11$. The FS has changed its quasi-stationary periodic character with a small oscillation amplitude and became quasi-stationary pseudoperiodic. Here the term "pseudoperiodic" is more precise than, e.g., "aperiodic" because there occur some "bursts" that are repeated at a certain interval for a long time and this interval changes with time. The foregoing concerns point 4: the FSs at points 2 and 1 (hereinafter referred to as FS-2 and FS-1, etc.) had already reached the steady state by $N = 200$ and 1000, respectively, and the FS-3 oscillation amplitude does not exceed 5%. But the deviations of FS-4 from the mean are considerable, up to 20–30%, and cannot be called "oscillations." In our opinion, here FS-4 makes attempts to make a transition to another branch of the solution, but a certain mechanism "returns" it to the first branch. The variants of

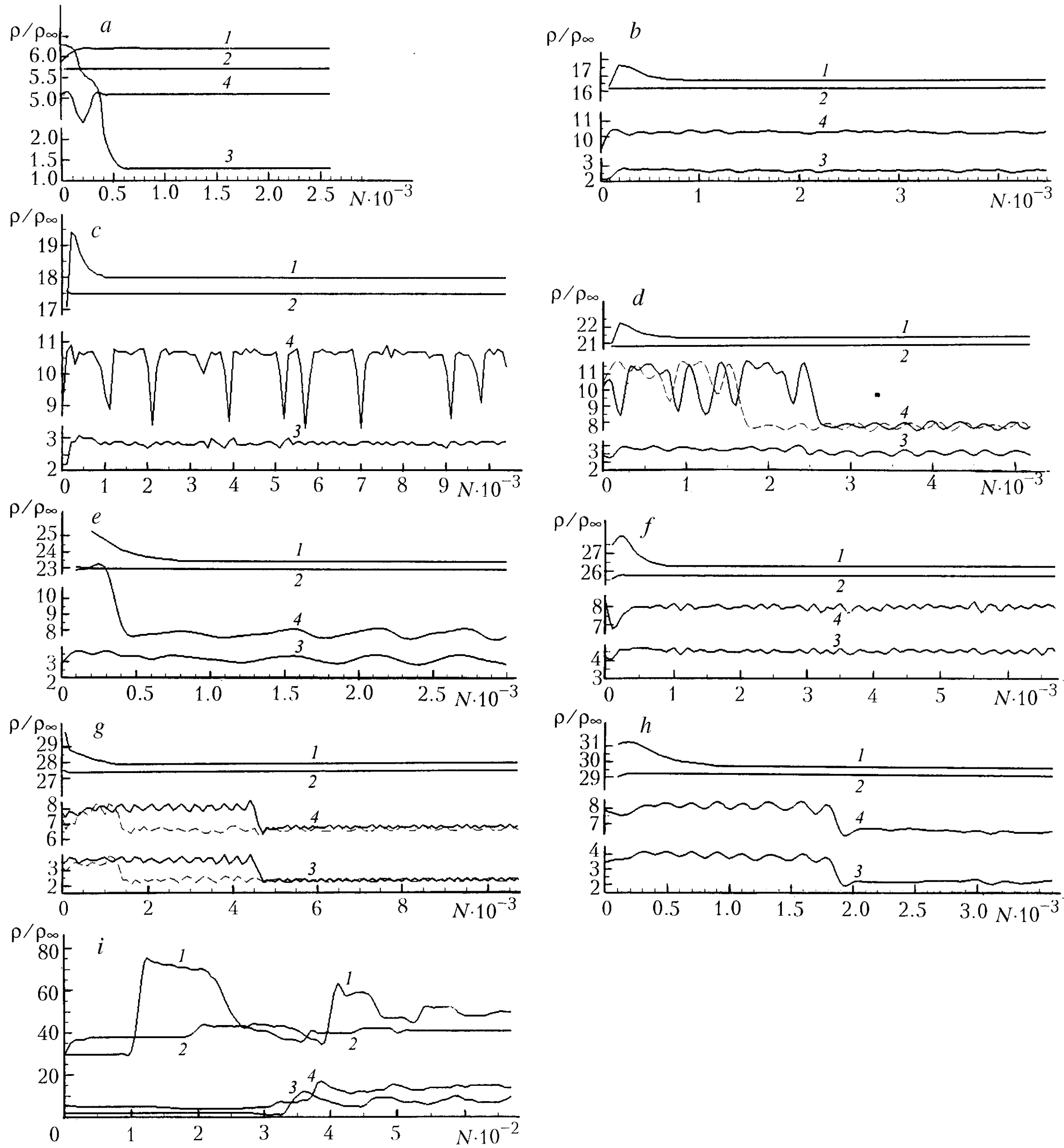

Fig. 2. Trajectories of advancement of solutions at reference points 1–4 (curves 1–4, respectively) from the starting conditions γ_0 to the final solution γ_s: a) from 1.67 to 1.4; b) 1.2 to 1.11; c) from 1.1 to 1.10; d) from 1.1 to 1.08; e) from 1.08 to 1.07; f) from 1.08 to 1.06; g) from 1.06 to 1.055; h) from 1.07 to 1.05; i) from 1.05 to 1.03.

the experiment for $\lambda_s = 1.10$ have shown the following. The algorithmic variants (computing mesh, iteration step) led to the same, in principle, RASs: RAS-1, -2, -3 were practically indistinguishable from one another, and all RASs-4, differing in details, had one and the same characteristic feature — an "outburst" of parameters at certain intervals, and their positions thereby correlated fairly well with one another. Note that the level of initial disturbances determined by the difference of γ_0 and γ_s, as mentioned above, also determined the possibility of revealing the probable transition of the solution from one branch to another. Therefore, in the computing experiment γ_0 should not differ considerably from γ_s, since from the theory of bifurcations it is fairly well known that a disturbed strongly oscillating field of parameters in the vicinity of the branching point may not enable the solution to go to any branch, leading to continuous and chaotic transitions from one branch to another (especially when the damping mechanism of oscillation energy dissipation is deficient or absent).

Figure 2d shows the RASs for the problem with $\gamma_s = 1.08$ and $\gamma_0 = 1.1$. In this experiment, we managed to reveal a bifurcation point in the hyperspace of diagnostic variables. We turn our attention to RAS-4: after a few "futile attempts" of the solution (at N = 150, 900, 1200, 1600, 2300) to go from one branch to another the transition, nevertheless, was made at N = 2700 (the value of the dimensionless time $N\tau = 2.7$) and FS-4, which was a quasi-stationary pseudoperiodic solution with a large amplitude, acquired the character of a quasi-stationary solution with a small (about 6%) oscillation amplitude. FS-1 and FS-2 thereby were still steadily stationary, and FS-3 had a weakly oscillatory character in the absolute "absence of the reaction" to the transition of FS-4. This shows that the general solution of the problem — the flow in the disturbed region — has changed in a specific way: the front shock angle in the lower (downstream) region of the flow has changed with the redistribution here of the gas-dynamic parameters. It will be recalled that equations of classical aerodynamics lead to a nonuniqueness of the solution of the problem of the supersonic flow inleakage on an obstacle (wedge, cone) with two possible values of the shock front angle. However, it is impossible to directly compare the classical solutions with the bifurcation solution under consideration because of the considerable inadequacy of the problems: nonuniformity of the supersonic flow of a blunt body and homogeneity (uniformity of the stream) of the classical problem.

The dashed line in Fig. 2d marks RAS-2 for the same problem with doubled τ. The transmission took place at N = 1500, i.e., at a value of the dimensionless time $N\tau = 3$, which correlates well (difference of 10%) with the previous result.

Figure 2e shows the RAS of the calculation of the problem with $\gamma_s = 1.07$, for which, as SCs, the FSs at $\gamma_0 = 1.08$ were taken from that branch of the solution which is characterized by a larger shock angle, i.e., before the transition to another branch (Fig. 2d). In this case, the transition of RAS-4 to "its" branch of the solution was made fairly quickly (at N = 300). Note that FS-3 and FS-4 have an oscillatory "almost" periodic character with a large wavelength ($\Delta N \approx 500$) with a fairly high (10–20%) oscillation amplitude. We turn our attention to RAS-1: unlike the previous variants of the problems, where all RASs-1 very quickly converged to a constant value, here this process was rather long and RAS-1 got to its asymptote even later than the transition in RAS-4 was made.

The experiment shown in Fig. 2f was performed in the "reverse" way: for the calculation with $\gamma_s = 1.06$, we used, as the SCs, the FSs with $\gamma_0 = 1.08$ from "their" branch of the solution. In this case, in the absence of the "necessity" of transition, all RASs smoothly adjusted themselves to the new FSs, which, while being quasi-stationary aperiodic, have a low (up to 5%) oscillation amplitude. However, as in the previous experiment, RAS-1 has the same prolonged time of reaching the steady state. This is a natural and expected result, since in high-speed flows disturbances are carried downstream (except for the boundary layer), and the level of disturbances, which is not too high in the leeward zone of the flow as well, practically did not affect the parameters in the windward zone.

The experiment illustrated in Fig. 2g had a longer iteration time. In this experiment, the transition of the solution from one branch to another was also predicted a priori. For the problem with $\gamma_s = 1.055$, we chose, as the SCs, the FS of the problem with $\gamma_0 = 1.06$ with a small $\gamma_s - \gamma_0$ difference so that the initial level of disturbances was not too high and did not provoke an oscillatory regime with a large amplitude in the vicinity of the bifurcation point, which could have led to numerous and chaotic transitions of the solution from one branch to another. It shows up as an unstable regime with a possibly unlimited growth of oscillations.

In this experiment, RAS-1 and RAS-2 very quickly brought the solutions to the steady state, and for RAS-3 and RAS-4, after a prolonged quasi-periodic regime (with a mean amplitude of ~5% in RAS-4 and 10% in RAS-3) a practically simultaneous (in the region of N = 4500) transition to another branch of the solution took place. Note that in the case of an early termination of this computing experiment the transition would not have been noticed, since the character of the solutions that had been obtained by this time, while being quasi-periodic but with a limited, small, and nonincreasing amplitude, gave no reason to predict the possibility of the transition.

Then, after the transition, RAS-3 and RAS-4 acquired a quasi-stationary character with oscillations at some mean line and amplitudes even smaller than before the transition. The positions of the mean lines before and after the transition changed for RAS-4 by 20% and for RAS-3 by 40%.

This experiment was complemented by starting the same problem with γ_s = 1.055 but with the SCs taken from the solution of the problem with γ_0 = 1.07 (of another branch). The results are presented in Fig. 2g by dashed lines. In this case, the level of initial disturbances was higher than in the main start (γ_s = 1.06). Therefore, the transition took place earlier (at N = 1300), although all the other mesh-algorithmic parameters were the same. The character of FS-3 and FS-4 was weakly oscillatory, analogous to the previous ones, with the same values of the level of the mean line. Note that in this experiment for the first time (at varied γ_s) the transition to RAS-3 appeared (before this it showed up only on RAS-4).

To analyze this phenomenon, we performed an experiment with a further decrease in γ_s down to 1.05. As SCs, we took the data from the first branch of the previous calculation (the use for SCs of the second, concluding branch of the solution led to expected results — the solution changed smoothly, staying on the same branch). The results given in Fig. 2h confirm the steady presence of the transition from one branch to another simultaneously in RAS-3 and RAS-4 (at N = 1900).

The last of the numerical experiments presented here is shown in Fig. 2i. To solve the problem with a very small value of γ_s = 1.03, we used the SCs of the solution of the problem with γ_0 = 1.05 from the second branch of the solution (on which the calculation shown in Fig. 2h was finished).

However, instead of expected RASs analogous to those obtained in the previous experiments of this set (with the oscillatory character of the existence on one branch of the solution also after some length of time, sometimes a very long one, of the transition to the second branch with a quasi-stationary regime and a low oscillation amplitude in RAS-3 and RAS-4 with a complete stationary picture of RAS-1 and RAS-2), a radically different result was obtained. Instead of the smooth line of the solution rearrangement, RAS-1 had a very stochastic form: an almost constant value on the initial segment, a sharp increase at time N = 100, then an almost horizontal "shelf," and, from time N = 200 on, a sharp but smooth decrease. Then such a character of the RAS-1 evolution was repeated several times (three cycles are visible) with a considerable decrease in the amplitude and by the time N = 700 RAS-1 becomes completely stationary — a straight line, as in the previous experiments. RAS-2 weakly follows these cycles of RAS-1 (with a more or less appreciable reaction only to the first of them) and then goes to the stationary solution. RAS-3 and RAS-4, too, have a different character than in the previous experiment: an almost nonoscillatory regime on the first length of time, the transition to "another branch" of the solution at time N = 200 for RAS-300 and N = 380 for RAS-4, and then the traditionally quasi-stationary behavior with small-amplitude oscillations near a certain constant mean line. Here the term "another branch" is put in quotes on purpose: first, the transition was realized with an increase in the numerical value of the solution, and all previous experiments had shown the transition from the first to the second branch only with a decrease in the traced value and never vice versa; second, the increase in the values in the downstream zone of the flow can simply be due to the nonstationary process of reorganization of the whole flow after the disappearance of the RAS-1 maximum.

To check the influence of the mesh parameters on the form of the solution obtained, the presented calculation was doubled several times with a gradual increase in the number of mesh nodes (by a factor of 100 in each coordinate direction), all other parameters of the computing experiment remaining unchanged. In so doing, the results of the initial calculation were repeated completely.

Discussion of the Problem: "Carbuncle" — Computational Effect or Physical Reality? By the "carbuncle" effect [9, 10] is meant the "swelling" of a part of the front shock arising in the windward zone of the flow, which steadily exists for a fairly long time. In general, this phenomenon is similar to the picture arising at a super- or a hypersonic flow of a blunt body by a gas flow with localized power supply (e.g., focused laser radiation [11]) to the incoming flow. In this case, the "carbuncle" effect can be explained in terms of the real physical process: the central part of the front shock is formed in the region of the nonuniform incoming flow, which leads to a "forward runaway" of a part of the front shock if the gas temperature before it is higher (and, accordingly, the density is lower). Note that such an effect can be attained by injecting gas (or plasma, which is more effective [12]) with the appearance of a so-called "aerodynamic needle," and the injection intensity determines the front shock configuration with a swelling of its central part in the direction of the mainstream incoming flow.

In the computer simulation of the hypersonic flow of blunt bodies, the computation algorithms enable one to calculate the "carbuncle" effect in an essentially different manner. Figure 3 shows the numerical solutions of one and the same problem carried out by several algorithms (for more details see [9]).

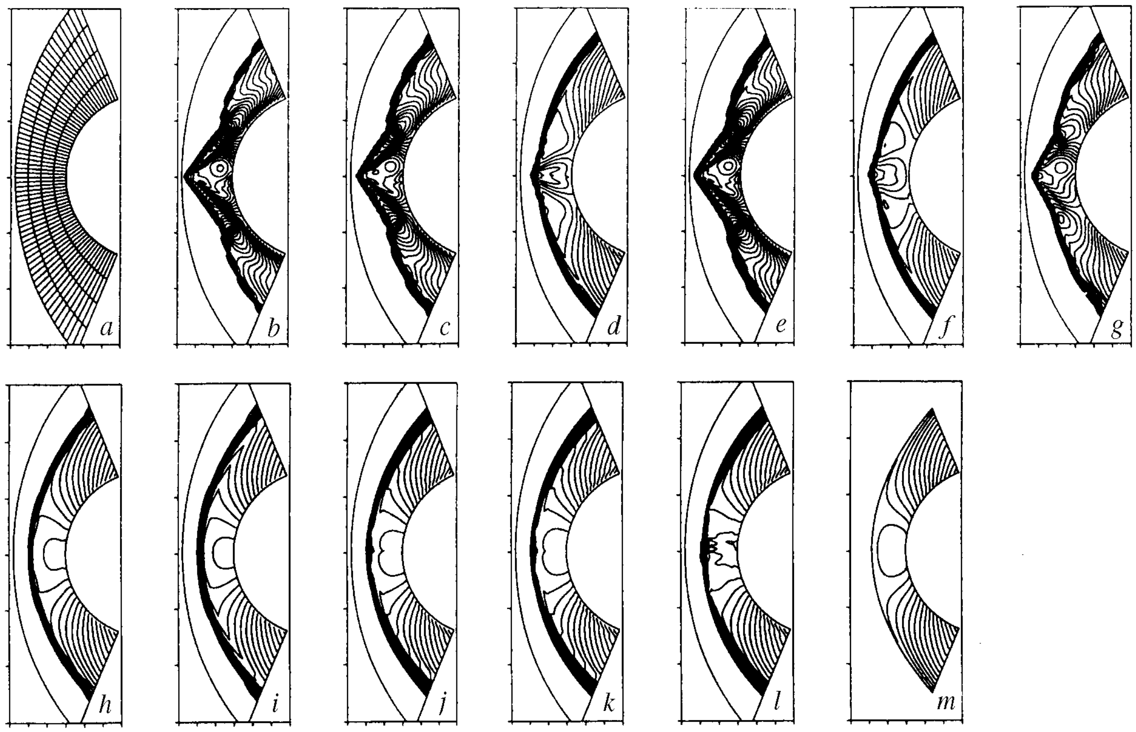

Fig. 3. Flow of blunt bodies by a hypersonic gas stream. Calculation mesh (a) and density isolines (b–n). Schemes with the presence of the "carbuncle" effect: FDSROG (b), FDSPAN (c), HUS (d), HLLC (e), AUSMV (f), AUSMD (g) and with its absence: FVSVL (h), HLL (i), AUSMV-VEL (j), AUSM-M (k), AUSM + (l), SHOFIT (m).

To the two-dimensional picture given in Fig. 3b (density isolines in the flow field) there corresponds the fragment of the one-dimensional curve of RAS-1 (given in Fig. 2i) with the maximum in the zone of the flow stagnation line that arose at time N = 100 and disappeared by time N = 250. In this experiment, as in [9, 10], there are no external factors capable of changing so strongly the form of the front wave, but, nevertheless, the "carbuncle" effect steadily takes place in a certain range of parameters (in this case, at very small values of the effective adiabatic index and high hypersonic Mach numbers). In the beginning of the supersonic range, at $M_\infty < 4$ this effect was not obtained at any values of γ even very close to its limiting value — unity.

It may be stated with a certain degree of caution that the existence domain of the "carbuncle" effect follows in the plane (M_∞, γ) the boundary of the strong instability domain [1, 4, 5] (in any case, it correlates well with it). In this domain, the external disturbance sources of the finite amplitude are capable of destroying the stationary flow pattern, leading to an unlimited growth of the solution, i.e., they provide, speaking in terms of the present work, the transition of the calculation from the stable branch of the solution to the unstable one.

In [9, 10], the situation with the "carbuncle" effect is somewhat different. The aim of the extensive investigation in [9] is to analyze various algorithms and the program codes realizing them, sorting according to the possibility of getting solutions with a strong (Fig. 3b, c, e, f) and a weak (Fig. 3d, f) "carbuncle" or without a "carbuncle" (Fig. 3h, i–n) on one and the same problem — a hypersonic (M_∞ = 20) flow of a blunt tip of a body. Debatable questions (we can agree to some of the arguments given in [9] but some of them raise objections) are beyond the scope of the present paper, but we must pose the principal question: are the "highly carbuncular schemes" of lower quality? And should one construct algorithms providing smoothness of the front shock at any price? What results will the use of such codes to calculate problems with power supply [11] or plasma injection [12] into the incoming flow yield? In our opinion, "careless" use of "highly stable" (in terms of [9], "carbuncle-free") algorithms can make it impossible to investigate problems in which instabilities that are due to physical causes take place and, the more so, problems of nonuniqueness of numerical solutions.

The importance of these problems will increase due to the development of multiprocessor computer systems and parallel counting algorithms [13], which will make it possible to go to a much higher level of mathematical modeling of physical gas dynamics problems.

This work was supported by the Russian Basic Research Foundation (projects 00-07-90297 and 02-01-00097).

NOTATION

M, Mach number; Re, Reynolds number of the incoming flow; Pr, Prandtl number; γ, effective adiabatic index of the gaseous medium; ρ, gas density; ω, degree of temperature dependence of the heat-conductivity coefficient; β, angle of the conical part of the body; R, radial coordinate counted from the center of the spherical blunting of the body; N, iteration number; τ, time step of the algorithm. Indices: 0, starting conditions of the calculation; s, conditions of current calculation (after the shock wave); ∞, values in the region of the incoming flow.

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